Review Notes on Maxwell's Equations

Review of Vector Products and the $\nabla$ Operator

The del, grad or nabla operator $\nabla$ is a vector, and can be part of a scalar product, a vector dot product or a vector cross product.

The product of this vector $\nabla$ with a scalar $\phi$ is another vector called the gradient of the scalar, $\nabla \phi$.

The dot product of two vectors is a scalar, and the dot product of $\nabla$ and another vector $\vec{A}$ is $\nabla \cdot \vec{A}$, a scalar called the divergence of the vector.

The cross product of two vectors is a vector normal to the plane of the two vectors, and the cross product of $\nabla$ and another vector $\vec{A}$ is $\nabla \times \vec{A}$, called the curl of the vector.

The dot product $\nabla \cdot \nabla$ is called the Laplacian, and is written $\nabla^2$.

See the inside covers of Pozar\(^1\) for useful summaries of vector operators (back cover), Maxwell's equations and transmission line relationships (front cover).

Review of Maxwell's Equations

Maxwell's equations, here in differential form, define the observed interaction among time-varying electric and magnetic fields, electric charge and the electric and magnetic characteristics of media.

<table>
<thead>
<tr>
<th>General form</th>
<th>Phasor form (cosinusoidal excitation)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\nabla \times \vec{E} = \frac{\partial \vec{B}}{\partial t}$</td>
<td>$\nabla \times \vec{E} = -j\omega \mu \vec{H}$</td>
</tr>
<tr>
<td>$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{j}$</td>
<td>$\nabla \times \vec{H} = j\omega \mu \vec{E}$</td>
</tr>
<tr>
<td>$\nabla \cdot \vec{D} = \rho = 0$ in charge-free regions</td>
<td></td>
</tr>
<tr>
<td>$\nabla \cdot \vec{B} = 0$</td>
<td></td>
</tr>
</tbody>
</table>

Field Interaction with Media

The additional polarization in dielectric materials resulting from interaction of applied electric field and the atomic or molecular structure results in the relationship \( \vec{D} = \varepsilon \vec{E} \), where \( \varepsilon \) may be complex \( \varepsilon = \varepsilon' - j\varepsilon'' \), the imaginary part accounting for loss.

In a material with conductivity \( \sigma \), a conduction current density will exist as \( \vec{J} = \sigma \vec{E} \).

In a magnetic material, an applied magnetic field may align magnetic dipole moments to produce a magnetic polarization resulting in the relationship \( \vec{B} = \mu \vec{H} \), where \( \mu \) may be complex \( \mu = \mu' - j\mu'' \), the imaginary part accounting for loss.

In non-isotropic media, \( \varepsilon \) and \( \mu \) may be tensors (matrices); additionally, some materials may be nonlinear.

The interaction of fields with media can be summarized in the relations here:

<table>
<thead>
<tr>
<th>Free space case</th>
<th>General dielectric, conductive or magnetic case</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \vec{D} = \varepsilon_0 \vec{E} ), ( \varepsilon_0 = 8.854 \times 10^{-12} \text{ F/m} )</td>
<td>( \vec{D} = \varepsilon \vec{E} )</td>
</tr>
<tr>
<td>( \vec{J} = \sigma \vec{E} )</td>
<td>( \vec{J} = \sigma \vec{E} )</td>
</tr>
<tr>
<td>( \vec{B} = \mu_0 \vec{H} ), ( \mu_0 = 4\pi \times 10^{-7} \text{ H/m} )</td>
<td>( \vec{B} = \mu \vec{H} )</td>
</tr>
</tbody>
</table>

Fields at Material Interfaces

The principal boundary conditions used in solving Maxwell's equation are at dielectric or conducting boundaries. These conditions are summarized here:

<table>
<thead>
<tr>
<th>Normal to interface</th>
<th>Tangential to interface</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dielectric Interface</td>
<td>Normal ( \vec{D} ) and ( \vec{B} ) continuous</td>
</tr>
<tr>
<td>Perfect Conductor</td>
<td>Normal ( \vec{D} = \rho ), ( \vec{B} = 0 )</td>
</tr>
<tr>
<td>Radiation Condition</td>
<td>Fields ( \rightarrow 0 ) for ( r \rightarrow \infty )</td>
</tr>
</tbody>
</table>
The Wave Equation

Taking the curl of the first equation, then substituting the second, and making use of the last two relationships plus a vector identity, derives a Helmholtz wave equation for $\mathbf{E}$

$$\nabla^2 \mathbf{E} + \omega^2 \mu \varepsilon \mathbf{E} = 0 \text{ and similarly can derive an identical one for } \mathbf{H}$$

$$\nabla^2 \mathbf{H} + \omega^2 \mu \varepsilon \mathbf{H} = 0$$

Plane Waves in a Lossless Medium

A solution of the wave equation exists that has only an $\hat{x}$ component, and for which the derivatives in the $x$ and $y$ directions are zero (uniform field in these directions). In this case the wave equation reduces to a simple, but illustrative, form

$$\frac{\partial^2 E_x}{\partial x^2} + k^2 E_x = 0$$

The solution to this type of equation is a wave described by

$$E_x(z) = E^+ e^{-jkz} + E^- e^{jkz}$$

with the propagation constant $k$ defined as

$$k = \frac{\omega}{\sqrt{\mu \varepsilon}} \text{ m}^{-1}$$

The velocity of wave propagation is

$$v_p = \frac{\omega}{k} = \frac{1}{\sqrt{\mu \varepsilon}}$$

The wavelength $\lambda$ is

$$\lambda = \frac{2\pi}{k} = \frac{2\pi v_p}{w} = \frac{v_p}{f}$$

The curl of time-varying $E_x$ gives rise to time-varying $H_y$ of the form

$$H_y(z) = H^+ e^{jkz} + H^- e^{jkz} = \frac{1}{\eta} (E^+ e^{-jkz} + E^- e^{jkz})$$

where $\eta = \sqrt{\mu / \varepsilon}$ is the wave impedance.

In the free-space case, $\eta_0 = \sqrt{\mu_0 / \varepsilon_0} = 377 \text{ ohms}$.

$H_x = H_z = 0$
In this case \( \vec{E} \) and \( \vec{H} \) are orthogonal to each other and to the direction of propagation \( \vec{z} \), so these waves, having only transverse components, are termed transverse electromagnetic (TEM) waves.

**Plane Waves in a Lossy Medium**

The same derivation applies in the case of lossy media (either \( \sigma \neq 0 \) or \( \varepsilon = \varepsilon' - j\varepsilon'' \), or both). The propagation constant becomes complex, signifying a decay term as well as the propagation term of the propagation constant.

A case of particular interest is that of a good conductor, defined as \( \sigma \gg \omega\varepsilon \). In this case a wave propagating into the material can only penetrate a depth called the skin depth, which is defined as

\[
\delta_s = \frac{2}{\omega\mu\sigma}.
\]

The skin depth varies as \( 1/\sqrt{f} \), and is of the order of \( 10^{-6} \) m at \( f = 10 \) GHz for most conductive metals such as aluminum, copper, gold and silver.

The solutions for general plane waves not necessarily propagating along a single axis have the same form as the simpler examples given above. It will be found that each geometric situation will result in characteristic forms of the solutions for waves that form a family of orthogonal modes defined by the boundary conditions.

The plane waves discussed above had their electric field vector pointing (polarized) in a fixed direction, and this is termed linear polarization. If we superpose two plane waves, one polarized in the \( \hat{x} \) direction and the other in the \( \hat{y} \) direction, traveling in the same direction, the linear combination of the two can form linear polarization at an angle or, if there is also a time (phase) difference, can form circular polarization in which the polarization vector rotates as the wave propagates. Similarly, two circularly polarized waves can be combined to form a linearly polarized wave.

Arbitrary field patterns can be expressed as a sum of orthogonal modes, which are the solutions of Maxwell's equations that fit the geometry of a particular class of components such as waveguide. We don't have to solve Maxwell's equations for every new problem, we just resolve the boundary conditions to determine the mode coefficients necessary to meet the boundary requirements.

**A Note on the History of Maxwell's Equations**

Maxwell's heroic contribution was the successful combination of the laws of electrostatics and magnetostatics so that they described accurately the phenomena of time-varying electromagnetic fields. The resulting mathematical descriptions of the interaction of electricity and magnetism, when combined, resulted in an easily recognized
wave equation predicting the existence of electromagnetic waves whose velocity of propagation in a vacuum would be

\[ v_p = \sqrt{\frac{1}{\mu_0\varepsilon_0}}, \text{ where } \mu_0 = 4 \times 10^{-7} \text{ H/m and } \varepsilon_0 = 8.85 \times 10^{-12} \text{ F/m.} \]

If you substitute these quantities into the equation for wave velocity, you find as Maxwell did that the predicted velocity is

\[ v = \sqrt{9 \times 10^{20}} = 3 \times 10^8 \text{ m/s.} \]

As he knew this was arguably close to the known velocity of light, this confirmed for Maxwell his earlier conjecture that light was electromagnetic in nature, and at the same time predicted electromagnetic radiation at other wavelengths was mathematically possible.

Excellent reviews of field and wave electromagnetics is found in Cheng\(^2\) and Inan\(^3\). These books contain a clear review of electrostatics, magnetostatics and how they underlie the waves that are the solutions of Maxwell's equations. Once a discipline is well-established, texts present it in a logical rather than historical order, so few modern texts give a picture of what was known when during the life of Maxwell himself. If you ever have a time for review and reflection, look for the wonderful Dover book, *The Scientific Papers of James Clerk Maxwell*.

Although the four equations are called Maxwell's equations, his genius was the postulation of the existence of the displacement current, which then resulted, in combination with the others, in a wave equation predicting electromagnetic waves that propagated at \( c = 1/\sqrt{\mu_0\varepsilon_0} \), the speed of light. From this he concluded correctly that light was electromagnetic in nature (he surmised this from the results of Faraday's experiments in polarization of light by magnetic interaction with dielectrics).

Many solutions of interest are in charge-free regions, but note that propagation through charged plasma\(^4\) represents a significant area of study.

We have the luxury of knowing how history worked out, and the study of electromagnetic fields and waves now takes for granted many of the great unknowns resolved by Maxwell's illustrious predecessors who are memorialized today in the units of electromagnetic quantities.

The physical description of the interaction of electricity, magnetism and matter began with the discovery of the phenomena and the mathematically stated laws of electrostatics. The scientific heroes of this period include names like Coulomb, Gauss, Faraday,

Ampere, Volterra and Ohm. Later, discovery of electrodynamic phenomena brought about the development of the more extensive set of relationships assembled by Maxwell, which themselves were polished further by the mathematicians who developed vector analysis as a shorthand to describe the equations of $\nabla \times \mathbf{E}$ electric and magnetic fields.